

Average subentropy and coherence of random mixed quantum states

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The generic aspects of the entanglement for random pure states are known to be established via the powerful phenomena of concentration of measure. Here, we find analytical expressions for the average subentropy over the set of random mixed states generated via various probability measures on it. The main ingredients in these results are Selberg's integrals. Surprisingly, our results show that the average subentropy of random mixed states approaches to the maximum value of the subentropy which is attained for the maximally mixed state as we increase the dimension. As an application, we find the average coherence of random mixed quantum states sampled from various probability measures, analytically. In the special case of the random mixed states sampled from the induced measure via the partial tracing of the random bipartite pure states, we establish the typicality of the relative entropy of coherence for random mixed states, numerically. In particular, we show that almost all random mixed states have relative entropy of coherence equal to the average relative entropy of coherence.

I. INTRODUCTION

The von Neumann entropy of a quantum system is of paramount importance in physics starting from thermodynamics [1, 2] to the quantum information theory, e.g., in studies of the classical capacity of a quantum channel and the compressibility of a quantum source [3], and serves as the least upper bound on the accessible information. The von Neumann entropy of an m dimensional density matrix ρ , is defined as $S(\rho) = -\sum_{j=1}^m \lambda_j \ln \lambda_j$, where $\lambda = \{\lambda_1, \dots, \lambda_m\}$ are eigenvalues of ρ . An analogous lower bound on the accessible information, obtained in Ref. [4] and called as the *subentropy* $Q(\rho)$, is defined as $Q(\rho) = -\sum_{i=1}^m \lambda_i^m \left(\prod_{j \neq i} (\lambda_i - \lambda_j) \right)^{-1} \ln \lambda_i$. Also, when two or more of the eigenvalues λ_j are equal, the value of Q is determined by taking a limit starting with unequal eigenvalues, unambiguously. The upper bound $S(\rho)$ and the lower bound $Q(\rho)$ on the accessible information are achieved for the ensemble of eigenstates of ρ and the Scrooge ensemble [4], respectively. Thus, the von Neumann entropy and the subentropy together define the range of the accessible information for a given density matrix. For a comparison between the von Neumann entropy and the subentropy, see Refs. [4–6].

On the other hand, miniaturization [7] and our ever increasing abilities to control systems at smaller and smaller scales motivate us to understand quantum concepts such as the coherence, entanglement and correlations, in general, from resource theoretical perspective [8–19]. In recent years, two inequivalent resource theories of coherence has been proposed [20–22] realizing the importance of the coherence as a resource in various physical situations. The operational meaning to the coherence quantifiers have been provided independently in Refs. [23] and [24]. Further, it has been proved that the coherence of a random pure state sampled from the uniform Haar measure is generic for higher dimensional systems,

i.e., most of the random pure states have the same amount of coherence [25]. The importance of this result and the similar results for entanglement of random bipartite pure states cannot be overemphasized. The average entanglement of random bipartite pure states, which is facilitated by the calculation of average entropy of the marginals of the random bipartite pure states [26–29], is proved typical [30]. This has resulted in various interesting consequences in quantum information theory [30–34], in the context of black holes [35] and in particular, in explaining the equal a priori probability postulate of statistical physics [36, 37]. But as we approach towards the realistic implementations of quantum technology, mixed states are encountered naturally due to the interaction between the system of interest and the external world. Therefore, consideration of average entanglement and coherence content of random mixed states is of great importance in realistic scenarios. However, to the best of our knowledge, there is no known result on the average coherence of random mixed states. This is partly because of the absence of the unique probability measure on set of mixed states unlike the unique Haar measure on the set of pure states.

In this work we aim at finding the average subentropy and relative entropy of coherence of random mixed states sampled from various induced measures including the one obtained via the partial tracing of the Haar distributed random bipartite pure states. We first find the exact expression for the average subentropy of random mixed states sampled from induced probability measures and importantly, show that the average subentropy appears naturally in the estimation of the average relative entropy of coherence of random mixed states. This facilitates the exact expression for the average coherence for random mixed states. It is important to emphasize here, since there is no unique measure on the set of mixed states, our results depend on the particular family of measures that we have considered in this paper. We further note that the subentropy is a nonlinear function of state and therefore, it is expected that the average subentropy of a random mixed state is not equal to the subentropy of the average state, which is the maximally mixed state. Surprisingly, we find that the average subentropy of a random mixed state approaches exponentially fast towards the maximum value of the subentropy, which is achieved for the maximally mixed state [4]. As an applica-

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tion of our results, we note that the average subentropy may also serve as the state independent quality factor for ensembles of states to be used for estimating accessible information. Interestingly, we find numerically that the average coherence of random mixed states, just like the average of random pure states, shows the concentration phenomena. This means that as we keep on increasing the dimension of the quantum system, more and more quantum states cluster around the average value of coherence that we obtain in our calculations. Our numerical results establish the typical nature of the relative entropy of coherence for random mixed states. In the similar spirit of generic nature of coherence for random pure states [25] which was established using the Lévy's lemma [38], we conjecture that it will be possible to establish the generic nature of average coherence of random mixed states using some generalized form of Lévy's lemma. We leave this for future works.

The paper is organized as follows. We start with a discussion of measures of coherence, random mixed states and some other necessary preliminaries in Sec. II. In Sec. III, we calculate the average subentropy of the random mixed states sampled from various induced probability measures on the set of mixed states. We then present our results on the average relative entropy of coherence of random mixed states in Sec. IV. Finally, we conclude in Sec. V with overview and implications of the results presented in the paper. The appendices show the explicit calculations of various integrals that appear in the main text.

II. QUANTUM COHERENCE AND INDUCED MEASURES ON THE SPACE OF MIXED STATES

Quantum coherence:— Various coherence monotones, that serve as the faithful measures of coherence [22, 39], are proposed based on the resource theory of coherence [22]. These monotones include the l_1 norm of coherence, relative entropy of coherence [22] and the geometric measure of coherence based on the geometric measure of entanglement [39]. In this work we consider only the relative entropy of coherence as a quantifier of coherence. Unless mentioned otherwise, by coherence we mean relative entropy of coherence throughout the paper.

Definition II.1. The relative entropy of coherence of a quantum state ρ , acting on an m -dimensional Hilbert space, is defined as [22]:

$$\mathcal{C}_r(\rho) := S(\rho || \Pi(\rho)) = S(\Pi(\rho)) - S(\rho), \quad (1)$$

where $\Pi(\rho) = \sum_{j=1}^m |j\rangle\langle j| \rho |j\rangle\langle j|$ for a fixed basis $\{|j\rangle : j = 1, \dots, m\}$ on the m -dimensional Hilbert space. $S(\rho)$ is the von Neumann entropy of ρ and is defined as $S(\rho) = -\text{Tr} \rho \ln \rho$. All the logarithms that appear in the paper are with respect to natural base.

Induced measures on the space of mixed states:— It is indispensable to define some probability measure μ on the space of density matrices in order to talk about the average properties of the random mixed states or the probability to choose a

mixed state with certain property. Unlike on the set of pure states, it is known that there exist several inequivalent measures on the set of density matrices, $\mathcal{D}(\mathbb{C}^m)$ (the set of trace one nonnegative $m \times m$ matrices). By the spectral decomposition theorem for Hermitian matrices, any density matrix ρ can be diagonalized by a unitary U . It seems natural to assume that the distributions of eigenvalues and eigenvectors of ρ are independent, implying μ to be product measure $\nu \times \mu_{\text{Haar}}$, where the measure μ_{Haar} is the unique Haar measure on the unitary group and measure ν , which defines the distribution of eigenvalues, but there is no unique choice for it [40, 41].

The induced measures on the $(m^2 - 1)$ -dimensional space $\mathcal{D}(\mathbb{C}^m)$ can be obtained by partial tracing the purifications $|\Psi\rangle$ in the larger composite Hilbert space of dimension mn and choosing the purified states according to the unique measure on it. Following Ref. [40], the joint density $P_{m(n)}(\lambda_1, \dots, \lambda_m)$ of eigenvalues $\Lambda = \{\lambda_1, \dots, \lambda_m\}$ of ρ , obtained via partial tracing, is given by

$$P_{m(n)}(\lambda_1, \dots, \lambda_m) = C_{m(n)} \delta \left(1 - \sum_{j=1}^m \lambda_j \right) \prod_{j=1}^m \lambda_j^{n-m} \theta(\lambda_j) |\Delta(\lambda)|^2, \quad (2)$$

where the theta function ensures that ρ is positive definite, $\Delta(\lambda) = \prod_{1 \leq i < j \leq m} (\lambda_i - \lambda_j)$ and $C_{m(n)}$ is the normalization constant. See Refs. [40, 41] for a good exposition of induced measures on the set of density matrices.

Now we introduce the family of integrals that will be a key to our work:

$$\begin{aligned} \mathcal{I}_m(\alpha, \gamma) &:= \int_0^\infty \dots \int_0^\infty \delta \left(1 - \sum_{j=1}^m \lambda_j \right) |\Delta(\lambda)|^{2\gamma} \prod_{j=1}^m \lambda_j^{\alpha-1} d\lambda_j, \\ &= \frac{1}{\Gamma(\alpha m + \gamma m(m-1))} \prod_{j=1}^m \frac{\Gamma(\alpha + \gamma(j-1)) \Gamma(1 + \gamma j)}{\Gamma(1 + \gamma)}, \end{aligned} \quad (3)$$

where $\alpha, \gamma > 0$, and $\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} dt$ is the Gamma function, defined for $\text{Re}(z) > 0$. The value of above family of integrals can be obtained using Selberg's integrals [40–42]. See appendix A for a quick review of Selberg's integrals. Let us define $C_m^{(\alpha, \gamma)} = 1/\mathcal{I}_m(\alpha, \gamma)$, which are called normalization constants. A family of probability measures over \mathbb{R}_+^m can be defined as:

$$\begin{aligned} d\nu_{\alpha, \gamma}(\lambda_1, \dots, \lambda_m) &:= C_m^{(\alpha, \gamma)} \delta \left(1 - \sum_{j=1}^m \lambda_j \right) |\Delta(\lambda)|^{2\gamma} \prod_{j=1}^m \lambda_j^{\alpha-1} d\lambda_j. \end{aligned} \quad (4)$$

Also, $\nu_{\alpha, \gamma}$ is a family of normalized probability measures over the probability simplex $\Delta_{m-1} := \{(\lambda_1, \dots, \lambda_m) \in \mathbb{R}_+^m : \sum_{j=1}^m \lambda_j = 1\}$, i.e.,

$$\nu_{\alpha, \gamma}(\Delta_{m-1}) = \int d\nu_{\alpha, \gamma}(\lambda_1, \dots, \lambda_m) = 1.$$

Now a family of probability measures $\mu_{\alpha,\gamma}$ over the set $D(\mathbb{C}^m)$ of all $m \times m$ density matrices over \mathbb{C}^m can be obtained via spectral decomposition of $\rho \in D(\mathbb{C}^m)$ with $\rho = U\Lambda U^\dagger$ as follows

$$d\mu_{\alpha,\gamma}(\rho) = d\nu_{\alpha,\gamma}(\Lambda) \times d\mu_{\text{Haar}}(U), \quad (5)$$

where $d\nu_{\alpha,\gamma}(\Lambda) = d\nu_{\alpha,\gamma}(\lambda_1, \dots, \lambda_m)$ and μ_{Haar} is the normalized uniform Haar measure. By definition, $\mu_{\alpha,\gamma}$ is a normalized probability measure over $D(\mathbb{C}^m)$. In the following, we will use this family of measures to calculate the average coherence of randomly chosen quantum states.

III. THE AVERAGE SUBENTROPY OF RANDOM MIXED QUANTUM STATES

Let us consider m dimensional random density matrices ρ sampled according to the family of product measures $\mu_{\alpha,\gamma}$, such that $d\mu_{\alpha,\gamma}(\rho) = d\nu_{\alpha,\gamma}(\Lambda) \times d\mu_{\text{Haar}}(U)$. The subentropy of a state ρ with the spectrum $\Lambda = \{\lambda_1, \dots, \lambda_m\}$ can be written as

$$\begin{aligned} Q(\Lambda) &= (-1)^{\frac{m(m-1)}{2}-1} \frac{\sum_{i=1}^m \lambda_i^m \ln \lambda_i \prod_{j \in \widehat{i}} \phi'(\lambda_j)}{|\Delta(\Lambda)|^2} \\ &= (-1)^{\frac{m(m-1)}{2}-1} \frac{1}{|\Delta(\Lambda)|^2} \frac{d}{dt} \left(\sum_{i=1}^m \lambda_i^t \prod_{j \in \widehat{i}} \phi'(\lambda_j) \right) \Big|_{t=m} \end{aligned} \quad (6)$$

where $\widehat{i} = \{1, \dots, m\} \setminus \{i\}$, $\phi'(\lambda_j) = \prod_{k \in \widehat{j}} (\lambda_j - \lambda_k)$ and $|\Delta(\Lambda)|^2 = |\Delta(\lambda_1, \dots, \lambda_m)|^2 = \left| \prod_{1 \leq i < j \leq m} (\lambda_i - \lambda_j) \right|^2$. The average subentropy over the set of mixed state is given by

$$\mathcal{I}_m^Q(\alpha, \gamma) = \int d\mu_{\alpha,\gamma}(\rho) Q(\rho) = \int d\nu_{\alpha,\gamma}(\Lambda) Q(\Lambda). \quad (7)$$

In the following and in later parts we make specific calculations only for the cases where $\gamma = 1$ and we are left with the family of probability measures, indexed by α . Now we have

$$\begin{aligned} \mathcal{I}_m^Q(\alpha, 1) &= (-1)^{\frac{m(m-1)}{2}-1} C_m^{(\alpha,1)} \int \left(\sum_{i=1}^m \lambda_i^m \ln \lambda_i \prod_{j \in \widehat{i}} \phi'(\lambda_j) \right) \delta \left(1 - \sum_{j=1}^m \lambda_j \right) \prod_{j=1}^m \lambda_j^{\alpha-1} d\lambda_j \\ &= (-1)^{\frac{m(m-1)}{2}-1} m C_m^{(\alpha,1)} \int \lambda_1^m \ln \lambda_1 \phi'(\lambda_2) \cdots \phi'(\lambda_m) \delta \left(1 - \sum_{j=1}^m \lambda_j \right) \prod_{j=1}^m \lambda_j^{\alpha-1} d\lambda_j \\ &= -m C_m^{(\alpha,1)} \sum_{k=0}^{m-1} (-1)^k \int_0^1 d\lambda_1 \lambda_1^{2(m-1)+\alpha-k} \ln \lambda_1 \int_0^\infty e_k \delta \left((1-\lambda_1) - \sum_{j=2}^m \lambda_j \right) |\Delta(\lambda_2, \dots, \lambda_m)|^2 \prod_{j=2}^m \lambda_j^{\alpha-1} d\lambda_j \\ &= -\frac{1}{m(m+\alpha-1)} \left[\sum_{k=0}^{m-1} (-1)^k [\psi(2(m-1)+\alpha+1-k) - \psi(m(m+\alpha-1)+1)] \right. \\ &\quad \times \left. \frac{\Gamma(2(m-1)+\alpha+1-k)}{\Gamma(k+1)\Gamma(m-k)\Gamma(m+\alpha-1-k)} \right], \end{aligned} \quad (8)$$

where in the second last line we have used $\phi'(\lambda_2) \cdots \phi'(\lambda_m) = (-1)^{\frac{m(m-1)}{2}} \phi'(\lambda_1) |\Delta(\lambda_2, \dots, \lambda_m)|^2$ and $\phi'(\lambda_1) = \sum_{k=0}^{m-1} (-1)^k e_k \lambda_1^{m-1-k}$ with $e_0 = 1$, $e_1 = \lambda_2 + \cdots + \lambda_m, \dots, e_{m-1} = \lambda_2 \cdots \lambda_m$, i.e., all e_k ($k = 0, 1, \dots, m-1$) are elementary symmetric polynomials in $\lambda_2, \dots, \lambda_{m-1}$, and the last line is obtained

from our calculation of the integral that we present in appendix B. In the remaining, we consider the *induced measure* $\mu_{m(n)}(m \leq n)$ over all the $m \times m$ density matrices of the m -dimensional quantum system via partial tracing over the n -dimensional ancilla of uniformly Haar-distributed random pure bipartite states of system and ancilla, which is as follows: for $\rho = U\Lambda U^\dagger$ with $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_m)$ and

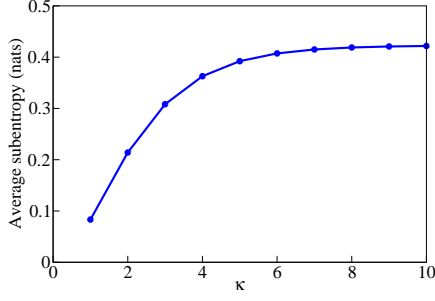


FIG. 1. (Color online) The average subentropy, obtained in Eq. (11), as a function of dimension $m = 2^\kappa$. Here x axis is dimensionless. Surprisingly, as we increase κ , the average subentropy approaches to the maximum value that subentropy can take.

$$U \in \mathcal{U}(m),$$

$$d\mu_{m(n)}(\rho) = d\nu_{m(n)}(\Lambda) \times d\mu_{\text{Haar}}(U), \quad (9)$$

where

$$d\nu_{m(n)}(\Lambda) = C_{m(n)} \delta \left(1 - \sum_{j=1}^m \lambda_j \right) |\Delta(\lambda)|^2 \prod_{j=1}^m \lambda_j^{n-m} d\lambda_j$$

is the joint distribution of eigenvalues $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_m)$ of the density matrix ρ , and $d\mu_{\text{Haar}}(U)$ is the uniform Haar measure over unitary group $\mathcal{U}(m)$. Apparently Eq. (9) is a special case of Eq. (5) when $(\alpha, \gamma) = (n - m + 1, 1)$. That is, $d\mu_{m(n)}(\rho) = d\mu_{n-m+1,1}(\rho)$ and $d\nu_{m(n)}(\Lambda) = d\nu_{n-m+1,1}(\Lambda)$. From this, we see that

$$\begin{aligned} & \mathcal{I}_m^Q(n - m + 1, 1) \\ &= -\frac{1}{mn} \sum_{k=0}^{m-1} (-1)^k \frac{\Gamma(m + n - k)}{\Gamma(k + 1)\Gamma(m - k)\Gamma(n - k)} \\ & \quad \times (\psi(m + n - k) - \psi(mn + 1)). \end{aligned} \quad (10)$$

If $m = n$, this situation corresponds to the measure induced by the Hilbert-Schmidt distance [40], then

$$\begin{aligned} \mathcal{I}_m^Q(1, 1) &= -\frac{1}{m^2} \sum_{k=0}^{m-1} (-1)^k (\psi(2m - k) - \psi(m^2 + 1)) \\ & \quad \times \frac{\Gamma(2m - k)}{\Gamma(k + 1)(\Gamma(m - k))^2}. \end{aligned} \quad (11)$$

We note that the closed form solutions for the average subentropy, given by Eqs. (8) and (11), for two particular probability measures, are very involved. Therefore, we turn to numerics to shed light on the behavior of the average subentropy, given by Eq. (11), for various values of m . In Table I we list the values of the average subentropy for various values of m . The Fig. 1 shows the graph of the average subentropy for random mixed states of dimension m as a function of m . We find that the average subentropy of a random mixed state approaches exponentially fast towards the maximum value of the subentropy, which is achieved for the maximally mixed

TABLE I. The average subentropy for random mixed states of dimension m . Δ is the difference between successive values in the second column and surprisingly shows an exponential convergence towards the maximum value of subentropy (≈ 0.42278) as we increase m . The difference between the successive values of the average subentropy is halved as we increase the number of qubits by one. See also Fig. (1).

m	$\mathcal{I}_m^{(1)}(1, 1)$ (nats)	Δ (nats)
2	0.083333	
4	0.214062	-0.130729
8	0.308176	-0.094114
16	0.362886	-0.054710
32	0.392185	-0.029299
64	0.407322	-0.015137
128	0.415012	-0.007690
256	0.418888	-0.003876
512	0.420833	-0.001945
1024	0.421808	-0.000975

state [4]. The maximum value of $Q(\rho)$ is approximately equal to 0.42278 [4]. This is surprising, since $Q(\rho)$ is a nonlinear function of ρ and it is not expected that the average subentropy should match with the subentropy of the average state, which is the maximally mixed state.

IV. THE AVERAGE COHERENCE OF RANDOM MIXED STATES AND TYPICALITY

In this section we calculate the average coherence of random mixed states and establish its typicality. Let $\rho = U\Lambda U^\dagger$ be a mixed full-ranked quantum state on \mathbb{C}^m with non-degenerate positive spectra $\lambda_j \in \mathbb{R}^+(j = 1, \dots, m)$, where $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_m)$. Then coherence of the state ρ (see Eq. (1)) is given by

$$\mathcal{C}_r(U\Lambda U^\dagger) = S(\Pi(U\Lambda U^\dagger)) - S(\Lambda).$$

The average coherence of the isospectral density matrices is defined as

$$\overline{\mathcal{C}}_r^{\text{iso}}(\Lambda) = \int d\mu_{\text{Haar}}(U) \mathcal{C}_r(U\Lambda U^\dagger),$$

and can be expressed in terms of the quantum subentropy, von Neumann entropy, and m -Harmonic number [43]:

$$\int d\mu_{\text{Haar}}(U) \mathcal{C}_r(U\Lambda U^\dagger) = H_m - 1 + Q(\Lambda) - S(\Lambda).$$

Here $Q(\Lambda)$ is the subentropy, given by Eq. (6), $S(\Lambda)$ is the von Neumann entropy of Λ and $H_m = \sum_{k=1}^m 1/k$ is the m -Harmonic number. From this, we see that the average coherence of isospectral full-ranked density matrices depends completely on the spectrum. Also, it is known that $0 \leq Q(\Lambda) \leq 1 - \gamma_{\text{Euler}} \approx 0.42278$, where γ_{Euler} is the Euler's constant. Now, using the product probability measures $d\mu_{\alpha, \gamma} = d\nu_{\alpha, \gamma} \times \mu_{\text{Haar}}(U)$, the average coherence of random

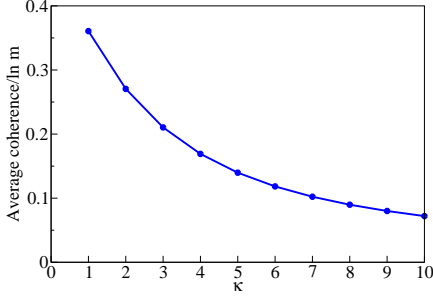


FIG. 2. (Color online) The plot shows the (scaled) average relative entropy of coherence, $\overline{\mathcal{C}}_r(1, 1)/\ln m$, obtained in Eq. (18), as a function of dimension $m = 2^k$. Both the axes are dimensionless.

mixed states is given by

$$\begin{aligned}
 \overline{\mathcal{C}}_r(\alpha, \gamma) &:= \int d\mu_{\alpha, \gamma}(\rho) \mathcal{C}_r(\rho) = \int d\mu_{\alpha, \gamma}(U \Lambda U^\dagger) \mathcal{C}_r(U \Lambda U^\dagger) \\
 &= \int d\nu_{\alpha, \gamma}(\Lambda) \left[\int d\mu_{\text{Haar}}(U) S(\Pi(U \Lambda U^\dagger)) - S(\Lambda) \right] \\
 &= H_m - 1 + \int d\nu_{\alpha, \gamma}(\Lambda) (Q(\Lambda) - S(\Lambda)) \\
 &= H_m - 1 + \mathcal{I}_m^Q(\alpha, \gamma) - \mathcal{I}_m^S(\alpha, \gamma), \tag{12}
 \end{aligned}$$

where

$$\mathcal{I}_m^Q(\alpha, \gamma) = \int d\nu_{\alpha, \gamma}(\Lambda) Q(\Lambda) \quad \text{and} \tag{13}$$

$$\mathcal{I}_m^S(\alpha, \gamma) = \int d\nu_{\alpha, \gamma}(\Lambda) S(\Lambda). \tag{14}$$

In the remaining, we again consider the *induced measure* $\mu_{m(n)}(m \leq n)$ over all the $m \times m$ density matrices of the m -dimensional quantum system via partial tracing over the n -dimensional ancilla of uniformly Haar-distributed random pure bipartite states of system and ancilla. In what follows, we calculate the above integrals in this case. From Eq. (8) of previous section, we know that, if $(\alpha, \gamma) = (n - m + 1, 1)$, then

$$\begin{aligned}
 \mathcal{I}_m^Q(n - m + 1, 1) &= -\frac{1}{mn} \sum_{k=0}^{m-1} (-1)^k \frac{\Gamma(m + n - k)}{\Gamma(k + 1) \Gamma(m - k) \Gamma(n - k)} \\
 &\quad \times (\psi(m + n - k) - \psi(mn + 1)). \tag{15}
 \end{aligned}$$

From the results of Page [26] and others [27–29] it is also known that

$$\mathcal{I}_m^S(n - m + 1, 1) = \sum_{k=n+1}^{mn} \frac{1}{k} - \frac{m-1}{2n}. \tag{16}$$

Using above two equations, we find that the average coherence of a random mixed state is given by

$$\begin{aligned}
 \overline{\mathcal{C}}_r(n - m + 1, 1) &= H_m - 1 - \frac{1}{mn} \sum_{k=0}^{m-1} (-1)^k \frac{\Gamma(m + n - k)}{\Gamma(k + 1) \Gamma(m - k) \Gamma(n - k)} (\psi(m + n - k) - \psi(mn + 1)) \\
 &\quad - \left(\sum_{k=n+1}^{mn} \frac{1}{k} - \frac{m-1}{2n} \right). \tag{17}
 \end{aligned}$$

Again for $m = n$, which corresponds to the measure induced by the Hilbert-Schmidt distance, we have

$$\overline{\mathcal{C}}_r(1, 1) = H_m - 1 - \frac{1}{m^2} \sum_{k=0}^{m-1} (-1)^k \frac{\Gamma(2m - k)}{\Gamma(k + 1) (\Gamma(m - k))^2} (\psi(2m - k) - \psi(m^2 + 1)) - \left(\sum_{k=m+1}^{m^2} \frac{1}{k} - \frac{m-1}{2m} \right). \tag{18}$$

We note again that the closed form solutions for the average coherence, given by Eqs. (17) and (18), for two particular probability measures, are very cumbersome. This is partly because of the apparent lack of the asymptotic expansion of the series in the two expressions, to the best of our knowledge. Therefore, we calculate numerical values of the average coherence, given by Eq. (18), for various values of m , to gain physical insights about our results. The numerical results are presented in Table II. Fig. 2 shows the graph of the (scaled) average relative entropy of coherence for random mixed states

as a function of dimension m . At this point, just like in the case of random pure states where it is proved that the average coherence is generic property of all the random pure states [25], one may hope to get the average relative entropy of the random mixed states to be the generic property of all the random mixed states. We have explored along this direction numerically. Our Fig. 3 shows that the average relative entropy of coherence is indeed a generic property of all the random mixed states. This is because as we increase the dimension of the density matrix, almost all the density matrices generated

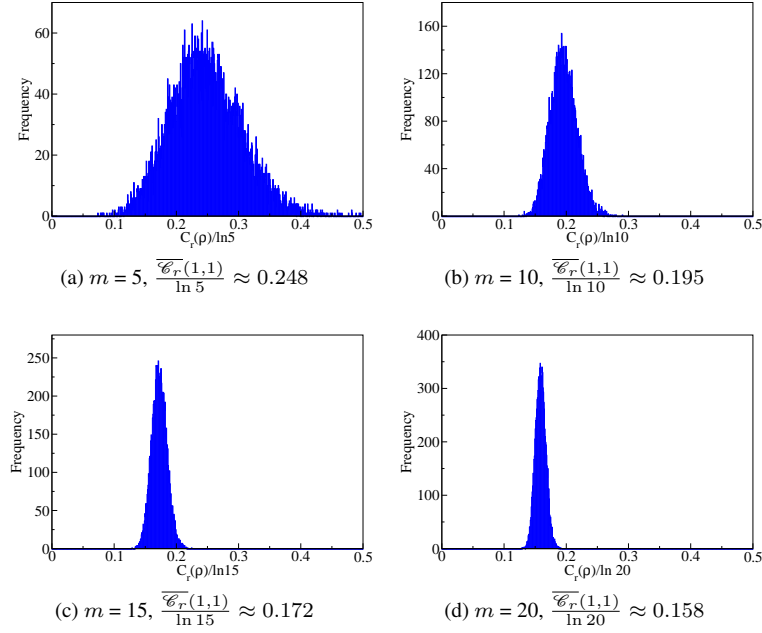


FIG. 3. (Color online) The frequency plot showing the (scaled) relative entropy of coherence, $C_r(\rho)/\ln m$ for random mixed states for dimensions $m = 2, 5, 10$ and 20 obtained via partial tracing of bipartite $m \times m$ Haar distributed pure states. Here x and y axes both are dimensionless. We note that as we increase the dimension the figure shows that more and more states have coherence close to a fixed value which is very close to the average value of coherence that we have calculated.

TABLE II. The (scaled) average relative entropy of coherence for random mixed states as a function dimension m . Δ , the difference between successive values in the second column, shows a rather slow convergence as a function of m , leading to the conclusion that the (scaled) average relative entropy of coherence for random mixed states may not settle to a fixed nonzero number.

m	$\overline{\mathcal{C}_r(1,1)}/\ln m$	Δ
2	0.360673	
4	0.270505	0.090168
8	0.210393	0.060112
16	0.169065	0.041328
32	0.139761	0.029304
64	0.118346	0.021415
128	0.102244	0.016102
256	0.089816	0.012428
512	0.079993	0.009823
1024	0.072064	0.007929

randomly show coherence approximately equal to the average relative entropy of coherence, given by Eq. (18). Thus, the average coherence of a random mixed state can be viewed as the typical coherence content of random mixed states. Moreover, in contrast to the case of pure states, average coherence of the random mixed states sampled from the measure, Eq. (5) with $\alpha = 1$ and $\gamma = 1$, decreases as we increase the dimension. Fig. 4 captures this behavior. This also shows that on an average random mixed states have less coherence compared to random pure states, as expected.

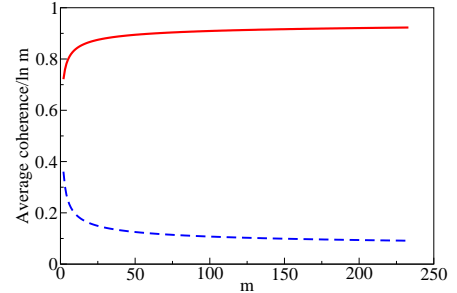


FIG. 4. (Color online) The (scaled) average relative entropy of coherence, $\mathcal{C}_r(1,1)/\ln m$, obtained in Eq. (18), and scaled average relative entropy of coherence for random pure states, which is given by $H_m - 1$ [25], as a function of dimension m . Both the axes are dimensionless. The dashed blue line and the solid red line represent the (scaled) average coherence for mixed and pure random states, respectively.

V. CONCLUSION

To conclude, we have provided analytical expressions for the average subentropy and the average relative entropy of coherence over the whole set of density matrices distributed according to the family of probability measures obtained via the spectral decomposition. Our calculations are facilitated by Selberg's integrals. Since there is no unique probability measure on the set of mixed quantum states, we emphasize that our results depend on the probability measures on the

set of mixed states, that we have considered. Since the exact expressions for the average subentropy and the average coherence are very complicated, we have resorted to some numerics for gaining physical insights. We find that as we increase the dimension of the quantum system, the average subentropy approaches towards the maximum value of subentropy (attained for the maximally mixed state) exponentially fast, which is surprising as the subentropy is a nonlinear function of density matrix. However, the scaled average coherence does not converge quickly to some fixed value unlike that for pure states. Interestingly, we numerically find that the coherence of random mixed states sampled from induced measures via the partial tracing show the concentration phenomena, indicating towards the generic nature of coherence content of random mixed states. Since quantum coherence is deemed as a useful resource for implementations of various quantum

technologies, our results will serve as a benchmark to gauge the resourcefulness of a generic mixed state for certain task at hand. Furthermore, our results may have some applications in black hole physics as to how much coherence can be there in the Hawking radiation for non-thermal states [35]. It may be worth thinking if the phenomena of concentration of measure, in particular Lévy's lemma, can be invoked to rigorously establish what we have found numerically. We leave this question for the future explorations.

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Appendix A: Selberg's Integrals and its consequences

Proposition A.1 (Selberg's Integrals, [42]). *If m is a positive integer and α, β, γ are complex numbers such that*

$$\operatorname{Re}(\alpha) > 0, \quad \operatorname{Re}(\beta) > 0, \quad \operatorname{Re}(\gamma) > -\min \left\{ \frac{1}{m}, \frac{\operatorname{Re}(\alpha)}{m-1}, \frac{\operatorname{Re}(\beta)}{m-1} \right\},$$

then

$$\begin{aligned} S_m(\alpha, \beta, \gamma) &= \int_0^1 \cdots \int_0^1 \left(\prod_{j=1}^m x_j^{\alpha-1} (1-x_j)^{\beta-1} \right) |\Delta(x)|^{2\gamma} [dx] \\ &= \prod_{j=1}^m \frac{\Gamma(\alpha + \gamma(j-1)) \Gamma(\beta + \gamma(j-1)) \Gamma(1 + \gamma j)}{\Gamma(\alpha + \beta + \gamma(m+j-2)) \Gamma(1 + \gamma)}, \end{aligned} \quad (\text{A1})$$

where $\Delta(x) = \prod_{1 \leq i < j \leq m} (x_i - x_j)$ and $[dx] = \prod_{j=1}^m dx_j$. Furthermore, if $1 \leq k \leq m$, then

$$\int_0^1 \cdots \int_0^1 \left(\prod_{j=1}^k x_j \right) \left(\prod_{j=1}^m x_j^{\alpha-1} (1-x_j)^{\beta-1} \right) |\Delta(x)|^{2\gamma} [dx] = S_m(\alpha, \beta, \gamma) \prod_{j=1}^k \frac{\alpha + \gamma(m-j)}{\alpha + \beta + \gamma(2m-j-1)}. \quad (\text{A2})$$

The following two integrals (Propositions A.2 and A.3) are direct consequences of Proposition A.1.

Proposition A.2 ([42]). *With the same conditions on the parameters α, γ ,*

$$\int_0^\infty \cdots \int_0^\infty |\Delta(x)|^{2\gamma} \prod_{j=1}^m x_j^{\alpha-1} e^{-x_j} dx_j = \prod_{j=1}^m \frac{\Gamma(\alpha + \gamma(j-1)) \Gamma(1 + \gamma j)}{\Gamma(1 + \gamma)}. \quad (\text{A3})$$

Proposition A.3 ([42]). *With the same conditions on the parameters α, γ , and $1 \leq k \leq m$,*

$$\int_0^\infty \cdots \int_0^\infty \left(\prod_{j=1}^k x_j \right) |\Delta(x)|^{2\gamma} \prod_{j=1}^m x_j^{\alpha-1} e^{-x_j} dx_j = \left(\prod_{j=1}^k (\alpha + \gamma(m-j)) \right) \left(\prod_{j=1}^m \frac{\Gamma(\alpha + \gamma(j-1)) \Gamma(1 + \gamma j)}{\Gamma(1 + \gamma)} \right). \quad (\text{A4})$$

In the following, we get Propositions A.4 and A.5 from Propositions A.2 and A.3, respectively, using Laplace transform.

Proposition A.4 ([40, 41]). *It holds that*

$$\begin{aligned} \frac{1}{C_m^{(\alpha, \gamma)}} &:= \int_0^\infty \cdots \int_0^\infty \delta \left(1 - \sum_{j=1}^m x_j \right) |\Delta(x)|^{2\gamma} \prod_{j=1}^m x_j^{\alpha-1} dx_j \\ &= \frac{1}{\Gamma(\alpha m + \gamma m(m-1))} \prod_{j=1}^m \frac{\Gamma(\alpha + \gamma(j-1)) \Gamma(1 + \gamma j)}{\Gamma(1 + \gamma)}. \end{aligned} \quad (\text{A5})$$

Proof. Let

$$F(t) := \int_0^\infty \cdots \int_0^\infty \delta \left(t - \sum_{j=1}^m x_j \right) |\Delta(x)|^{2\gamma} \prod_{j=1}^m x_j^{\alpha-1} dx_j.$$

Performing Laplace transform ($t \rightarrow s$) to $F(t)$ gives

$$\begin{aligned} \tilde{F}(s) &= \int_0^\infty F(t) e^{-st} dt \\ &= \int_0^\infty \cdots \int_0^\infty \exp \left(-s \sum_{j=1}^m x_j \right) |\Delta(x)|^{2\gamma} \prod_{j=1}^m x_j^{\alpha-1} dx_j \\ &= s^{-\alpha m - 2\gamma \binom{m}{2}} \int_0^\infty \cdots \int_0^\infty |\Delta(y)|^{2\gamma} \prod_{j=1}^m y_j^{\alpha-1} e^{-y_j} dy_j, \end{aligned}$$

leading to the following via inverse Laplace transform ($s \rightarrow t$) to $\tilde{F}(s)$:

$$F(t) = \frac{t^{\alpha m + \gamma m(m-1) - 1}}{\Gamma(\alpha m + \gamma m(m-1))} \int_0^\infty \cdots \int_0^\infty |\Delta(x)|^{2\gamma} \prod_{j=1}^m x_j^{\alpha-1} e^{-x_j} dx_j,$$

Therefore,

$$\frac{1}{C_m^{(\alpha, \gamma)}} = F(1) = \frac{1}{\Gamma(\alpha m + \gamma m(m-1))} \times \int_0^\infty \cdots \int_0^\infty |\Delta(x)|^{2\gamma} \prod_{j=1}^m x_j^{\alpha-1} e^{-x_j} dx_j.$$

Hence the desired identity via Eq. (A3). \square

Proposition A.5. *It holds that, for $1 \leq k \leq m$,*

$$\begin{aligned} & \int_0^\infty \cdots \int_0^\infty \left(\prod_{j=1}^k x_j \right) \delta \left(1 - \sum_{j=1}^m x_j \right) |\Delta(x)|^{2\gamma} \prod_{j=1}^m x_j^{\alpha-1} dx_j \\ &= \frac{1}{\Gamma(\alpha m + \gamma m(m-1) + k)} \int_0^\infty \cdots \int_0^\infty \left(\prod_{j=1}^k x_j \right) |\Delta(x)|^{2\gamma} \prod_{j=1}^m x_j^{\alpha-1} e^{-x_j} dx_j. \end{aligned} \quad (\text{A6})$$

Proof. Similarly, let

$$f(t) := \int_0^\infty \cdots \int_0^\infty \left(\prod_{j=1}^k x_j \right) \delta \left(t - \sum_{j=1}^m x_j \right) |\Delta(x)|^{2\gamma} \prod_{j=1}^m x_j^{\alpha-1} dx_j.$$

Then,

$$\begin{aligned} \tilde{f}(s) &= \int_0^\infty \cdots \int_0^\infty \left(\prod_{j=1}^k x_j \right) \exp \left(- \sum_{j=1}^m s x_j \right) |\Delta(x)|^{2\gamma} \prod_{j=1}^m x_j^{\alpha-1} dx_j \\ &= s^{-(\alpha m + \gamma m(m-1) + k)} \int_0^\infty \cdots \int_0^\infty \left(\prod_{j=1}^k y_j \right) |\Delta(y)|^{2\gamma} \prod_{j=1}^m y_j^{\alpha-1} e^{-y_j} dy_j. \end{aligned}$$

Therefore,

$$f(t) := \frac{t^{\alpha m + \gamma m(m-1) + k - 1}}{\Gamma(\alpha m + \gamma m(m-1) + k)} \int_0^\infty \cdots \int_0^\infty \left(\prod_{j=1}^k y_j \right) |\Delta(y)|^{2\gamma} \prod_{j=1}^m y_j^{\alpha-1} e^{-y_j} dy_j.$$

By setting $t = 1$ in the above equation, we derived the desired identity via Eq. (A4). \square

Proposition A.6. *It holds that*

$$\frac{d}{dt} \left(\frac{\Gamma(t+a)}{\Gamma(t+b)} \right) = (\psi(t+a) - \psi(t+b)) \frac{\Gamma(t+a)}{\Gamma(t+b)}, \quad (\text{A7})$$

where $\psi(t) = \frac{d}{dt} \ln \Gamma(t)$.

Appendix B: The calculation of integrals in the main text

Denote $\phi(x) := \prod_{j=1}^m (x - x_j)$. Then $\phi'(x) = \sum_{i=1}^m \prod_{j \in \hat{i}} (x - x_j)$. Thus $\phi'(x_i) = \prod_{j \in \hat{i}} (x_i - x_j)$. Furthermore, we have

$$\prod_{i=1}^m \phi'(x_i) = \prod_{i=1}^m \prod_{j \in \hat{i}} (x_i - x_j) = (-1)^{\frac{m(m-1)}{2}} |\Delta(x)|^2. \quad (\text{B1})$$

Here $|\Delta(x)|^2 = |\Delta(x_1, \dots, x_m)|^2$ is called the *discriminant* of ϕ [42]. If we expand the polynomial $\phi(x)$, then we have: let $e_0 \equiv 1$,

$$\phi(x) = x^m - \left(\sum_{j=1}^m x_j \right) x^{m-1} + \dots + (-1)^m \prod_{j=1}^m x_j = \sum_{j=0}^m (-1)^j e_j x^{m-j}, \quad (\text{B2})$$

where $e_j (j = 1, \dots, m)$ is the j -th elementary symmetric polynomial in x_1, \dots, x_m .

In what follows, we calculate the integral $\mathcal{I}_m^Q(\alpha, \gamma)$ for $\gamma = 1$. Propositions A.4 and A.5 will be used frequently for $\gamma = 1$.

$$\begin{aligned} & \mathcal{I}_m^Q(\alpha, 1) \\ &= -m C_m^{(\alpha, 1)} \sum_{k=0}^{m-1} (-1)^k \int_0^1 d\lambda_1 \lambda_1^{2(m-1)+\alpha-k} \ln \lambda_1 \int_0^\infty \dots \int_0^\infty e_k \delta \left((1 - \lambda_1) - \sum_{j=2}^m \lambda_j \right) |\Delta(\lambda_2, \dots, \lambda_m)|^2 \prod_{j=2}^m \lambda_j^{\alpha-1} d\lambda_j. \end{aligned}$$

It suffices to calculate a family of integrals in terms of the following form: for $k = 0, 1, \dots, m-1$,

$$\int_0^\infty \dots \int_0^\infty e_k \delta \left((1 - \lambda_1) - \sum_{j=2}^m \lambda_j \right) |\Delta(\lambda_2, \dots, \lambda_m)|^2 \prod_{j=2}^m \lambda_j^{\alpha-1} d\lambda_j.$$

If $k = 0$, then

$$\begin{aligned} & \int_0^\infty \dots \int_0^\infty e_0 \delta \left((1 - \lambda_1) - \sum_{j=2}^m \lambda_j \right) |\Delta(\lambda_2, \dots, \lambda_m)|^2 \prod_{j=2}^m \lambda_j^{\alpha-1} d\lambda_j \\ &= (1 - \lambda_1)^{(m-1)(m+\alpha-2)-1} \int_0^\infty \delta \left(1 - \sum_{j=1}^{m-1} x_j \right) |\Delta(x_1, \dots, x_{m-1})|^2 \prod_{j=1}^{m-1} x_j^{\alpha-1} dx_j \\ &= (1 - \lambda_1)^{(m-1)(m+\alpha-2)-1} \frac{\prod_{j=1}^{m-1} \Gamma(\alpha + j - 1) \Gamma(1 + j)}{\Gamma((m-1)(m+\alpha-2))}. \end{aligned} \quad (\text{B3})$$

Here we used Proposition A.4 in the last equality.

If $1 \leq k \leq m-1$, it suffices to calculate the following:

$$\begin{aligned} & \int_0^\infty \dots \int_0^\infty \left(\prod_{j=1}^k \lambda_{j+1} \right) \delta \left((1 - \lambda_1) - \sum_{j=2}^m \lambda_j \right) |\Delta(\lambda_2, \dots, \lambda_m)|^2 \prod_{j=2}^m \lambda_j^{\alpha-1} d\lambda_j \\ &= (1 - \lambda_1)^{(m-1)(m+\alpha-2)+k-1} \times \\ & \quad \int_0^\infty \dots \int_0^\infty \left(\prod_{j=1}^k x_j \right) \delta \left(1 - \sum_{j=1}^{m-1} x_j \right) |\Delta(x_1, \dots, x_{m-1})|^2 \prod_{j=1}^{m-1} x_j^{\alpha-1} dx_j \\ &= (1 - \lambda_1)^{(m-1)(m+\alpha-2)+k-1} \frac{\prod_{j=1}^{m-1} \Gamma(\alpha + j - 1) \Gamma(1 + j)}{\Gamma((m-1)(m+\alpha-2) + k)} \frac{\Gamma(m + \alpha - 1)}{\Gamma(m + \alpha - 1 - k)}. \end{aligned} \quad (\text{B4})$$

Here we used Proposition A.5. Next, we calculate:

$$\int_0^1 d\lambda_1 \lambda_1^t \int_0^\infty \dots \int_0^\infty e_k \delta \left((1 - \lambda_1) - \sum_{j=2}^m \lambda_j \right) |\Delta(\lambda_2, \dots, \lambda_m)|^2 \prod_{j=2}^m \lambda_j^{\alpha-1} d\lambda_j.$$

(1). If $k = 0$, then

$$\begin{aligned}
& \int_0^1 d\lambda_1 \lambda_1^t \int_0^\infty \cdots \int_0^\infty \delta \left((1 - \lambda_1) - \sum_{j=2}^m \lambda_j \right) |\Delta(\lambda_2, \dots, \lambda_m)|^2 \prod_{j=2}^m \lambda_j^{\alpha-1} d\lambda_j \\
&= \frac{\prod_{j=1}^{m-1} \Gamma(\alpha + j - 1) \Gamma(1 + j)}{\Gamma((m-1)(m + \alpha - 2))} \times \int_0^1 \lambda_1^t (1 - \lambda_1)^{(m-1)(m + \alpha - 2) - 1} d\lambda_1 \\
&= \frac{\prod_{j=1}^{m-1} \Gamma(\alpha + j - 1) \Gamma(1 + j)}{\Gamma((m-1)(m + \alpha - 2))} \times \frac{\Gamma(t+1) \Gamma((m-1)(m + \alpha - 2))}{\Gamma(t+1 + (m-1)(m + \alpha - 2))} \\
&= \frac{\Gamma(t+1) \prod_{j=1}^{m-1} \Gamma(\alpha + j - 1) \Gamma(1 + j)}{\Gamma(t+1 + (m-1)(m + \alpha - 2))}.
\end{aligned}$$

By taking the derivative with respect to t on both sides, we get

$$\begin{aligned}
& \int_0^1 d\lambda_1 \lambda_1^t \ln \lambda_1 \int_0^\infty \cdots \int_0^\infty \delta \left((1 - \lambda_1) - \sum_{j=2}^m \lambda_j \right) |\Delta(\lambda_2, \dots, \lambda_m)|^2 \prod_{j=2}^m \lambda_j^{\alpha-1} d\lambda_j \\
&= [\psi(t+1) - \psi(t+1 + (m-1)(m + \alpha - 2))] \frac{\Gamma(t+1) \prod_{j=1}^{m-1} \Gamma(\alpha + j - 1) \Gamma(1 + j)}{\Gamma(t+1 + (m-1)(m + \alpha - 2))}.
\end{aligned}$$

For $t = 2(m-1) + \alpha$, we have

$$\begin{aligned}
& \int_0^1 d\lambda_1 \lambda_1^{2(m-1)+\alpha} \ln \lambda_1 \int_0^\infty \cdots \int_0^\infty \delta \left((1 - \lambda_1) - \sum_{j=2}^m \lambda_j \right) |\Delta(\lambda_2, \dots, \lambda_m)|^2 \prod_{j=2}^m \lambda_j^{\alpha-1} d\lambda_j \\
&= [\psi(2(m-1) + \alpha + 1) - \psi(m(m + \alpha - 1) + 1)] \frac{\Gamma(2(m-1) + \alpha + 1) \prod_{j=1}^{m-1} \Gamma(\alpha + j - 1) \Gamma(1 + j)}{\Gamma(m(m + \alpha - 1) + 1)}.
\end{aligned}$$

(2). If $1 \leq k \leq m-1$, then

$$\begin{aligned}
& \int_0^1 d\lambda_1 \lambda_1^t \int_0^\infty \cdots \int_0^\infty e_k \delta \left((1 - \lambda_1) - \sum_{j=2}^m \lambda_j \right) |\Delta(\lambda_2, \dots, \lambda_m)|^2 \prod_{j=2}^m \lambda_j^{\alpha-1} d\lambda_j \\
&= \binom{m-1}{k} \int_0^1 d\lambda_1 \lambda_1^t \int_0^\infty \cdots \int_0^\infty \left(\prod_{j=1}^k \lambda_{j+1} \right) \delta \left((1 - \lambda_1) - \sum_{j=2}^m \lambda_j \right) |\Delta(\lambda_2, \dots, \lambda_m)|^2 \prod_{j=2}^m \lambda_j^{\alpha-1} d\lambda_j \\
&= \binom{m-1}{k} \frac{\prod_{j=1}^{m-1} \Gamma(\alpha + j - 1) \Gamma(1 + j)}{\Gamma((m-1)(m + \alpha - 2) + k)} \prod_{j=1}^k (m + \alpha - j - 1) \times \int_0^1 \lambda_1^t (1 - \lambda_1)^{(m-1)(m + \alpha - 2) + k - 1} d\lambda_1 \\
&= \binom{m-1}{k} \frac{\prod_{j=1}^{m-1} \Gamma(\alpha + j - 1) \Gamma(1 + j)}{\Gamma((m-1)(m + \alpha - 2) + k)} \prod_{j=1}^k (m + \alpha - j - 1) \times \frac{\Gamma(t+1) \Gamma((m-1)(m + \alpha - 2) + k)}{\Gamma(t+1 + (m-1)(m + \alpha - 2) + k)} \\
&= \binom{m-1}{k} \frac{\Gamma(t+1) \prod_{j=1}^{m-1} \Gamma(\alpha + j - 1) \Gamma(1 + j)}{\Gamma(t+1 + (m-1)(m + \alpha - 2) + k)} \prod_{j=1}^k (m + \alpha - j - 1) \\
&= \binom{m-1}{k} \frac{\Gamma(t+1) \prod_{j=1}^{m-1} \Gamma(\alpha + j - 1) \Gamma(1 + j)}{\Gamma(t+1 + (m-1)(m + \alpha - 2) + k)} \frac{\Gamma(m + \alpha - 1)}{\Gamma(m + \alpha - 1 - k)}.
\end{aligned}$$

By taking the derivative with respect to t , we get

$$\begin{aligned}
& \int_0^1 d\lambda_1 \lambda_1^t \ln \lambda_1 \int_0^\infty \cdots \int_0^\infty e_k \delta \left((1 - \lambda_1) - \sum_{j=2}^m \lambda_j \right) |\Delta(\lambda_2, \dots, \lambda_m)|^2 \prod_{j=2}^m \lambda_j^{\alpha-1} d\lambda_j \\
&= \binom{m-1}{k} [\psi(t+1) - \psi(t+1 + (m-1)(m + \alpha - 2) + k)] \\
&\quad \times \frac{\Gamma(t+1) \prod_{j=1}^{m-1} \Gamma(\alpha + j - 1) \Gamma(1 + j)}{\Gamma(t+1 + (m-1)(m + \alpha - 2) + k)} \frac{\Gamma(m + \alpha - 1)}{\Gamma(m + \alpha - 1 - k)}.
\end{aligned}$$

For $t = 2(m-1) + \alpha - k$, we have

$$\begin{aligned}
& \int_0^1 d\lambda_1 \lambda_1^{2(m-1)+\alpha-k} \ln \lambda_1 \int_0^\infty \cdots \int_0^\infty e_k \delta \left((1-\lambda_1) - \sum_{j=2}^m \lambda_j \right) |\Delta(\lambda_2, \dots, \lambda_m)|^2 \prod_{j=2}^m \lambda_j^{\alpha-1} d\lambda_j \\
&= \binom{m-1}{k} [\psi(2(m-1) + \alpha - k + 1) - \psi(m(m+\alpha-1) + 1)] \\
&\quad \times \frac{\Gamma(2(m-1) + \alpha - k + 1) \prod_{j=1}^{m-1} \Gamma(\alpha + j - 1) \Gamma(1 + j)}{\Gamma(m(m+\alpha-1) + 1)} \frac{\Gamma(m+\alpha-1)}{\Gamma(m+\alpha-1-k)}.
\end{aligned}$$

In summary, we get

$$\begin{aligned}
& \mathcal{I}_m^Q(\alpha, 1) \\
&= -mC_m^{(\alpha,1)} \left[\int_0^1 d\lambda_1 \lambda_1^{2(m-1)+\alpha} \ln \lambda_1 e_0 \int_0^\infty \cdots \int_0^\infty \delta \left((1-\lambda_1) - \sum_{j=2}^m \lambda_j \right) |\Delta(\lambda_2, \dots, \lambda_m)|^2 \prod_{j=2}^m \lambda_j^{\alpha-1} d\lambda_j \right. \\
&\quad \left. + \sum_{k=1}^{m-1} (-1)^k \int_0^1 d\lambda_1 \lambda_1^{2(m-1)+\alpha-k} \ln \lambda_1 \int_0^\infty \cdots \int_0^\infty e_k \delta \left((1-\lambda_1) - \sum_{j=2}^m \lambda_j \right) |\Delta(\lambda_2, \dots, \lambda_m)|^2 \prod_{j=2}^m \lambda_j^{\alpha-1} d\lambda_j \right] \\
&= -mC_m^{(\alpha,1)} \left[\sum_{k=0}^{m-1} (-1)^k \binom{m-1}{k} [\psi(2(m-1) + \alpha - k + 1) - \psi(m(m+\alpha-1) + 1)] \right. \\
&\quad \left. \times \frac{\Gamma(2(m-1) + \alpha - k + 1) \prod_{j=1}^{m-1} \Gamma(\alpha + j - 1) \Gamma(1 + j)}{\Gamma(m(m+\alpha-1) + 1)} \frac{\Gamma(m+\alpha-1)}{\Gamma(m+\alpha-1-k)} \right] \\
&= -\frac{mC_m^{(\alpha,1)} \Gamma(m+\alpha-1) \prod_{j=1}^{m-1} \Gamma(\alpha + j - 1) \Gamma(1 + j)}{\Gamma(m(m+\alpha-1) + 1)} \\
&\quad \times \sum_{k=0}^{m-1} (-1)^k \binom{m-1}{k} [\psi(2(m-1) + \alpha + 1 - k) - \psi(m(m+\alpha-1) + 1)] \frac{\Gamma(2(m-1) + \alpha + 1 - k)}{\Gamma(m+\alpha-1-k)} \\
&= -\frac{1}{m(m+\alpha-1)} \left[\sum_{k=0}^{m-1} (-1)^k [\psi(2(m-1) + \alpha + 1 - k) - \psi(m(m+\alpha-1) + 1)] \right. \\
&\quad \left. \times \frac{\Gamma(2(m-1) + \alpha + 1 - k)}{\Gamma(k+1)\Gamma(m-k)\Gamma(m+\alpha-1-k)} \right].
\end{aligned}$$
